MATH 121A Prep: Bases

1. Show that $\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ form a basis for $\mathbb{R}^2$. Generalize this to a basis for $\mathbb{R}^n$. Conclude that $\mathbb{R}^n$ has dimension n. [Note: This is called the standard basis for $\mathbb{R}^n$.]

Solution: First we show $\vec{e_1}$ and $\vec{e_2}$ are linearly independent. Suppose $c_1\vec{e_1} + c_2\vec{e_2} = \vec{0}$, then

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = c_1\vec{e_1} + c_2\vec{e_2} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

The equality of these vectors means $c_1 = 0$ and $c_2 = 0$ as desired. Now we show they span $\mathbb{R}^2$. Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be any vector in $\mathbb{R}^2$. Then

$$v_1\vec{e_1} + v_2\vec{e_2} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{v}$$

as we wanted. Since these vectors are linearly independent and span $\mathbb{R}^2$, they are a basis for $\mathbb{R}^2$. We can generalize this to a basis for $\mathbb{R}^n$ by considering vectors

$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \qquad \cdots, \qquad \vec{e_n} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

That is, we have vectors $\vec{e_1}, \ldots, \vec{e_n}$ where $\vec{e_i}$ is a vector with a 1 in the ith row and zeros elsewhere. This is a basis for $\mathbb{R}^n$ by a very similar proof to the $\mathbb{R}^2$ case. Therefore $\mathbb{R}^n$ has dimension n.

2. Prove that the vectors $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$ do not span $\mathbb{R}^3$.

Solution: We need only show that the matrix with these vectors as columns row reduces to one with a row of all zeros.

$$\begin{bmatrix} 0 & 2 & 4 \\ 1 & 1 & 1 \\ -1 & 1 & 3 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ -1 & 1 & 3 \end{bmatrix} \xrightarrow{R3=R3+R1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 2 & 4 \end{bmatrix} \xrightarrow{R3=R3-R2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a row of all zeros these vectors cannot span $\mathbb{R}^3$.

3. Show that two vectors are linearly dependent if and only if one is a scalar multiple of the other.

Solution: First suppose that vectors $\vec{v_1}$ and $\vec{v_2}$ are linearly dependent. Then there are constants c_1 and c_2 not both zero so that $c_1\vec{v_1} + c_2\vec{v_2} = \vec{0}$.

Case 1: $c_1 \neq 0$. Then $\vec{v_1} = -c_2/c_1\vec{v_2}$ is a scalar multiple of $\vec{v_2}$.

Case 2: $c_2 \neq 0$. Then $\vec{v_2} = -c_1/c_2\vec{v_1}$ is a scalar multiple of $\vec{v_1}$.

In either case one vector is a scalar multiple of the other.

Now assume $\vec{v_1}$ and $\vec{v_2}$ are vectors and one is a scalar multiple of the other.

Case 1: $\vec{v_1}$ is a multiple of $\vec{v_2}$. Then there is a real number c so $\vec{v_1} = c\vec{v_2}$. Then $\vec{v_1} - c\vec{v_2} = \vec{0}$ so they are linearly dependent.

Case 2: $\vec{v_2}$ is a multiple of $\vec{v_1}$. Then there is a real number c so that $\vec{v_2} = c\vec{v_1}$. So $c\vec{v_1} - \vec{v_2} = \vec{0}$ and they are linearly dependent.

In either case the vectors are linearly dependent.